Quantum field ground state of the sine-Gordon model with finite soliton density: exact results

1982 J. Phys. A: Math. Gen. 15 507

(http://iopscience.iop.org/0305-4470/15/2/021)

J. Phys. A: Math. Gen. **15** (1982) 507–525. Printed in Great Britain

Quantum fluid ground state of the sine-Gordon model with finite soliton density: exact results

F D M Haldane†

Institut Laue-Langevin, 156X, 38042 Grenoble, France

Received 29 April 1981

Abstract. The Bethe ansatz solutions of the massive Thirring model are renormalised and used to obtain exactly the ground-state properties of the sine-Gordon model with a finite density of topological solitons. Using a theory of 1D quantum fluids, the correlation function exponents and low-energy excitation spectrum are obtained. Expansions in various limits are given, and various crossovers described.

1. Introduction

In recent years, there has been intensive study of the sine-Gordon (SG) boson quantum field theory in one space dimension ('1 + 1 dimensions') and the equivalent fermion massive Thirring model (MTM) (Coleman 1975). The model is also of interest in connection with quasi-one-dimensional materials with commensurate and incommensurate density waves at low temperatures. The model has turned out to have a factorisable S-matrix (Zamolodchikov and Zamolodchikov 1979) and has recently been solved by the Bethe ansatz (Bergknoff and Thacker 1979). In this paper I use these solutions to derive the ground-state properties of the model in the *quantum fluid state* where it has a finite density of topological defects or *solitons*. The parameters describing the low-energy excitations of such a system, as well as the exponents controlling the power-law fall-off of ground-state correlations, are explicitly presented. Crossover behaviour in various limits is described.

The two variants of the model have Lagrangian densities

$$\mathscr{L}^{\mathrm{SG}} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + (m_0/\beta)^2 \cos(\beta\phi), \tag{1.1}$$

$$\mathscr{L}^{\mathrm{MTM}} = \bar{\psi}(\mathrm{i}\gamma_\mu \partial^\mu - m_0)\psi - \tfrac{1}{2}g(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) \qquad -\pi < g. \tag{1.2}$$

The relation between the two dimensionless coupling parameters β and g is

$$(4\pi/\beta^2) = 1 + g/\pi > 0. \tag{1.3}$$

To define the model, a high-energy cut-off is required; the bare mass m_0 must undergo multiplicative renormalisation in the field-theoretical limit where this cut-off goes to infinity and Lorentz invariance is re-established. This limit, in which the SG and MTM become equivalent (Coleman 1975), only exists for $0 < \beta^2 \leq 8\pi$, $-\frac{1}{2}\pi < g$. Outside this range, the MTM has no mass gap, even when m_0 is finite, and a Lorentz-invariant field

† Address from September 1981: Department of Physics, University of Southern California, Los Angeles, CA 90007, USA.

0305-4470/82/020507 + 19$02.00

theory cannot be obtained when m_0 is non-zero. The limiting case $\beta^2 = 8\pi$ must be obtained by letting $\beta^2 \to 8\pi$ *after* taking the field-theoretical limit.

In the region $0 < \beta^2 \leq 8\pi$ discussed here, the mass term is relevant. A treatment of the infrared divergent perturbation expansion in the mass term m_0 by a scaling theory leads to the idea of universal crossover functions describing the crossover from the unstable (massless) weak coupling fixed point to the stable strong coupling fixed point, out of the range of convergence of the perturbation expansions. It would seem to be of interest to extract such crossover functions from the available exact Bethe ansatz solutions. Such a calculation implies an identification of the renormalised coupling with some physical quantity calculable within the Bethe ansatz formalism.

The approach described here is based on a recent theory of one-dimensional quantum fluids (Haldane 1980, 1981a). The SG theory has a topologically conserved charge, the *soliton number* N_s, corresponding to the fermion charge of the MTM. In the classical limit $\beta^2 \to 0$ of the SG, the system with a finite density of solitons has a 'solid' soliton lattice ground state, but for any finite value of the quantum parameter β^2, zero-point fluctuations liquify this lattice, and the ground state is a 1D quantum fluid. Such a fluid shows critical behaviour at $T = 0$; its correlations decay asymptotically with characteristic power laws. The exponents are controlled by a single dimensionless parameter which will be designated by φ, following Haldane (1980). In the classical limit, the '*quantum fluctuation parameter*' $\exp(\varphi)$ goes to zero. In the semiclassical limit, $\exp(\varphi)$ is essentially a measure of the ratio of the amplitude of the zero-point fluctuations of a particle of the fluid in the 'cage' formed by its neighbours, compared with the mean interparticle separation. In the dilute soliton limit, the quantum fluctuation parameter $\exp(\varphi)$ has the limiting value 1, characteristic of dilute fluid systems which are dominated by the 'hard core' interaction between particles. This limit is perhaps best exemplified by the gas of free spinless fermions, where the effect of the Pauli principle is equivalent to a hard core. On the other hand, as will be shown here, in the *concentrated* soliton limit, where the effect of the mass term in (1.2) becomes negligible, $\exp(\varphi)$ tends to the limit appropriate to the corresponding mass*less* Thirring model, in fact $\exp(2\varphi) \to \beta^2/4\pi$. In this work, the variation of this dimensionless ground-state parameter as a function of soliton density will be taken to be the principal crossover function of the SG/MTM system. Recent developments in the theory of 1D quantum fluids (Haldane 1980, 1981b) have finally shown how the value of this parameter can be extracted from Bethe ansatz solutions.

The calculation is given in a fully renormalised form, in terms of the soliton mass m_s, and the 'velocity of light' c, which characterises the Lorentz invariance of the field theory. A number of different length scales appear in the model: the principal one is the *quantum soliton length* $R_s = \hbar/m_s c$. For $\beta^2 < 4\pi$, there are also excitonic soliton–antisoliton bound states, or 'breathers', with masses $m_b^{(n)} = 2m_s \sin(\frac{1}{2}\pi\kappa n)$, $n < \kappa^{-1}$, where

$$\kappa = \frac{(\beta^2/8\pi)}{1-(\beta^2/8\pi)} = \frac{(1+2g/\pi)}{(2+g/\pi)}, \qquad 0 \leq \kappa \leq \infty, \tag{1.4}$$

(this quantity κ will be adopted as the standard parametrisation of the SG/MTM coupling in this work). For $\beta^2 < 4\pi$, or $\kappa < 1$, a second important length scale is the *principal breather length* $R_b = \hbar/m_b^{(1)}c$. As will be noted here, it is this quantum breather length R_b that becomes the characteristic width of the classical soliton in the limit $\beta^2 \to 0$, $\kappa \to 0$. Connection with the Frank–van der Merwe (1949) solution for the ground-state energy of the classical soliton lattice will be made; this classical soliton width R_b controls the crossover between the high- and low-density classical soliton lattice regimes. In addition, as $\beta^2 \to 0$, it will be shown that there is a *third* length scale $\bar{R} = 2R_b \ln[2\exp(C)/\pi\kappa] \gg R_b$ which controls the crossover between the dilute quantum fluid and the dilute semiclassical fluid corresponding to the semiclassical extension of the Frank–van der Merwe classical behaviour. Similarly, the limiting model with $\beta^2 = 8\pi$, $\kappa = \infty$ has qualitatively different behaviour in the *high* soliton density limit to models with finite κ, and a length scale $R^* = \text{constant} \times R_s(\frac{1}{2}\pi\kappa)^{-1/2} \exp(-\frac{1}{2}\kappa)$ characterises the crossover. This crossover regime is also accessible by straightforward perturbation theory in the MTM mass.

2. The soliton fluid state: general considerations

The excitation spectrum (for fixed charge) of the SG/MTM model about a ground state with finite soliton density can be understood from examination of the case $\kappa = 1$, corresponding to the *free fermion limit* of the MTM. The ground state is non-degenerate; the excitation spectrum can be described in terms of four 'elementary' components, and is schematically depicted in figure 1. 'Type I' excitations take a particle from the Fermi surface to high energies, and cover the whole range of momenta, except $P = 0$; the

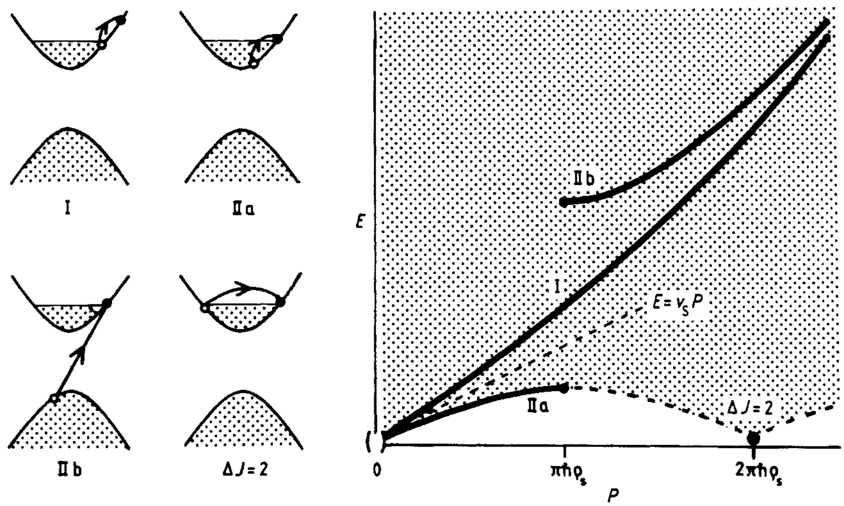

Figure 1. Fixed-charge excitation spectrum of the soliton fluid state of the quantum sine-Gordon or massive Thirring model. (Left): construction of the 'elementary' type I, type IIa, type IIb and current ($\Delta J = 2$) excitations in the free fermion limit ($\beta^2 = 4\pi$ or $\kappa = 1$) of the MTM. (Right): energy against momentum diagram for the excitation spectrum; for $\beta^2 < 4\pi$ ($\kappa < 1$) the type IIb branch should be identified with the excitonic bound state or 'breather'. The energy of the elementary current excitation $J = 0 \to J = 2$ is $2\pi\hbar v_J/L$; as $\beta^2 \to 0$ ($\kappa \to 0$) $v_J \to 0$ and the ground state becomes degenerate with crystalline order. The type IIa and type IIb excitations become acoustical and optical phonons, while the type I excitations become dynamical solitons or 'single-particle' modes. The shaded region marks the continuum of the full excitation spectrum. The $q = 0$ collective modes are 'missing': their place is taken by excitations that change the topological charge but leave the system in a ground state. The quantum fluid behaviour described here involves only long-wavelength acoustical modes with sound velocity v_S and current excitations. Together with the charge excitations, these fully span the space of low-energy states.

group velocity is the sound velocity at long wavelengths, but rises asymptotically to the light velocity at high energies. 'Type IIa' excitations involve excitation from an occupied particle state to the nearest point on the Fermi surface, and are restricted to the momentum range $|P| < \pi\hbar\rho_s$, $P \neq 0$. 'Type IIb' excitations take a particle from an occupied 'negative energy' state in the filled Dirac sea to a state at the Fermi level; these excitations are restricted to the momentum range $|P| > \pi\hbar\rho_s$. Finally, *current* excitations carrying momentum in units of $2\pi\hbar\rho_s$ are made by excitation of a particle from the highest filled state on the one side of the Fermi surface to the lowest empty state on the other side. This involves an energy $O(1/L)$ (where L is the periodic boundary length) which is quadratic in the total current. This catalogue can be extended to include the 'missing' $P=0$ modes as excitations that change the total soliton charge, but leave the system in the appropriate ground state. The combined type II states have a single gap at $P = \pm\pi\hbar\rho_s$; they can be regarded as a single set of modes. From an observation by Overhauser (1965), it is known that the Hilbert space can be fully spanned by the charge and current excitations, plus a single set of $q \neq 0$ collective modes that can be constructed as combinations of type I and type II excitations.

The excitations of the interacting system with $\kappa \neq 1$ soluble by the Bethe ansatz have a similar character, and the characterisation into type I and type II excitations follows the terminology introduced by Lieb (1963) for the Bose fluid. For $\kappa < 1$, the type IIb states may be taken to be the principal breather excitations, which are excluded from the range $|P| < \pi\hbar\rho_s$ by the Pauli principle. In the classical limit $\kappa \to 0$, the ground state develops long-range crystalline order, and becomes degenerate as the energy for making current excitations (now Umklapp processes where the lattice absorbs momentum) vanishes. In this limit, the type IIa excitations become the acoustical phonons, while the type IIb excitations are the optical phonons; the existence of a single gap in the phonon spectrum has been noted in the studies of the classical dynamics (Gupta and Sutherland 1976). In this limit it is possible to regard the type II excitations as the complete set of collective modes of the system; the type I excitations are then nonlinear combinations of these, and are the 'single-particle' or 'dynamic soliton' excitations. The fact that such excitations where a single particle of the fluid is given high momentum persist as a localised moving entity, and do not decay into delocalised collective radiation, is a particular characteristic of 'integrable' systems.

The above discussion has briefly described a number of interesting features of the excitation spectrum of the soliton fluid. However, the rest of this work will only be concerned with quite general quantum fluid aspects of the ground-state and low-energy excitation spectrum, unrelated to the presence or absence of integrability. Only the charge, current and long-wavelength acoustical modes are needed for this discussion.

According to Haldane (1980, 1981a, b, c), the low-energy spectrum of a 1D quantum fluid (without internal degrees of freedom such as spin) can be represented as

$$H = \hbar\Big(v_S \sum_{q\neq 0} |q| b_q^\dagger b_q + \tfrac{1}{2}(\pi/L)(v_N(N-N_0)^2 + v_J(J)^2)\Big), \tag{2.1}$$

$$P = \hbar\Big([k_F + (\pi/L)(N-N_0)]J + \sum_{q\neq 0} q b_q^\dagger b_q\Big), \qquad k_F = \pi\rho_s, \tag{2.2}$$

$$j = v_J(J/L), \tag{2.3}$$

$$(-1)^J = +1 \text{ (bosons)}, \; -(-1)^N \text{ (fermions)}. \tag{2.4}$$

Here $b_q^\dagger$, $qL/2\pi$ integral, are collective boson modes describing long-wavelength density fluctuations. N is the total charge, and J is an integral quantum number proportional to the quantised current, $j = v_J(J/L)$. For a given total charge, J is restricted to either even or odd values by the selection rule (2.4), which is the only reflection of the statistics of the component particles of the fluid that distinguishes the SG from the MTM. The elementary current excitations $J \to J+2$ carry a characteristic momentum $2\hbar k_F$, perhaps more familiar in connection with Fermi systems, but quite general to 1D quantum fluids. In the classical limit, $v_J \to 0$, and $2k_F$ becomes the reciprocal lattice vector of the solid ground state.

The Hamiltonian is characterised by three parameters with dimensions of velocity: v_S is the sound velocity, while v_N is related to the compressibility, and v_J to an effective kinetic mass density. A fundamental property of 1D quantum fluids is the relation

$$v_S = (v_N v_J)^{1/2}. \tag{2.5}$$

This has been explicitly verified for systems such as the SG/MTM that are solved by the Bethe ansatz (Haldane 1981b). This relation allows the 'quantum fluctuation parameter' $\exp(\varphi)$ to be defined through

$$v_N = v_S \exp(-2\varphi), \qquad v_J = v_S \exp(2\varphi). \tag{2.6}$$

As independently demonstrated by calculations based on (i) the Luttinger model (Haldane 1980, 1981a) and (ii) the 1D harmonic fluid (Haldane (1982), the form of the low-energy correlation functions is controlled by this parameter. The long-distance density–density correlations are the same for the SG and MTM. At $T = 0$,

$$\langle \rho_s(x)\rho_s(0)\rangle \sim \rho_s^2\Big(1 - \eta^{-1}(2k_F x)^{-2} + \sum_{m=1}^{\infty} A_m \cos(2mk_F x)|x\rho_s|^{-m^2\eta}\Big) \tag{2.7}$$

where $\eta = 2\exp(2\varphi)$. This is the general form for a 1D fluid; the A_m are model-dependent coefficients. The single-particle soliton correlations have the asymptotic behaviour

$$\langle \psi_s^\dagger(x)\psi_s(0)\rangle \sim \eta|x\rho_s|^{-1/\eta} \sum_{m=0}^{\infty} B_m \cos(2mk_F x)|x\rho_s|^{-m^2\eta} \qquad (\text{SG}),$$

$$\sim \eta|x\rho_s|^{-1/\eta} \sum_{m=0}^{\infty} C_m \sin[2(m+\tfrac{1}{2})k_F|x|]|x\rho_s|^{-(m+1/2)^2\eta} \qquad (\text{MTM}). \tag{2.8}$$

The difference between the SG and MTM results is due to the selection rule (2.4); boson single-particle correlations only involve even harmonics of $2k_F$, and fermion ones odd harmonics. The classical limit corresponds to $\exp(\varphi) \to 0$; the fluid then develops long-range order in its periodic density correlations, and the single-particle correlations vanish. For free spinless fermions, $\eta = 2$.

Calculation of v_S and $\exp(\varphi)$ suffices to establish the low-energy structure of a 1D quantum fluid. In fact, the *Lorentz invariance* of the SG/MTM system allows both these quantities to be related to the variation of the soliton chemical potential as a function of density ρ_s at $T = 0$. The ground state with finite momentum (i.e. finite current quantum number J, but no density wave excitations) can be obtained from the $P = 0$ ground state by a Lorentz transformation. Comparison with (2.1) allows v_J to be identified, while v_N is obtained from the compressibility. In an open system such as that under consideration, the time-like component of the relativistic vector which has space-like component cP is the *enthalpy*, $E_0 + pL$, where E_0 is the energy and $p = -\partial E_0/\partial L|_N$ is the pressure.

Since E_0 has the form $L\varepsilon_0(N_s/L)$, it is easily found that the enthalpy in the rest frame is given by $\mu_s N_s$, where μ_s is the chemical potential $\partial E_0/\partial N_s|_L$ for a system at rest. Making a Lorentz transformation with parameter $\gamma = \cosh^{-1}(1-v^2/c^2)^{-1/2}$ gives

$$E + pL = \mu_s N_s \cosh(\gamma), \qquad cP = \mu_s N_s \sinh(\gamma). \tag{2.9}$$

For small P, this leads to

$$E(P) = E_0 + \tfrac{1}{2}c^2P^2/\mu_s N_s + \ldots . \tag{2.10}$$

Comparison with (2.1) leads to the identification

$$v_J = \pi\hbar\rho_s c^2/\mu_s. \tag{2.11}$$

In the low soliton density limit $\rho_s \to 0$, $\mu_s \to E_s = m_s c^2$, the soliton rest energy, and $v_J \to \pi\hbar\rho_s/m_s$, a result characteristic of Galilean invariance. The quantity v_N is simply related to the compressibility; from (2.1),

$$v_N = (\pi\hbar)^{-1}\partial\mu_s/\partial\rho_s. \tag{2.12}$$

Combining these results,

$$v_S/c = (v_N v_J)^{1/2}/c = [(\rho_s/\mu_s)\partial\mu_s/\partial\rho_s]^{1/2}. \tag{2.13}$$

This is the result characteristic for a relativistic fluid. Similarly,

$$\exp(2\varphi) = (v_J/v_N)^{1/2} = \pi\hbar c[(\rho_s/\mu_s)\partial\rho_s/\partial\mu_s]^{1/2}. \tag{2.14}$$

The quantum nature of $\exp(\varphi)$ is apparent. It is convenient to express $\hbar c$ in terms of the soliton rest energy $E_s = m_s c^2$ and quantum soliton length $R_s = \hbar/m_s c$; then

$$\exp(2\varphi) = \pi E_s R_s(\partial\rho_s^2/\partial\mu_s^2)^{1/2}. \tag{2.15}$$

3. Derivation of the Bethe ansatz equations for the soliton fluid

In this section I will obtain the renormalised Bethe ansatz equations describing the properties of the ground state of the SG/MTM with finite soliton density. The renormalisation of the equations describing the excitations about this ground state can be carried out in an analogous way, but will not be given here.

Bergknoff and Thacker (1979) have given the MTM Bethe ansatz equations:

$$(R_s)_0^{-1}\sinh(\alpha_i) = 2\pi N_i/L + L^{-1}\sum_j \Theta(\alpha_i - \alpha_j),$$

$$E_0 = -(E_s)_0\sum_i \cosh(\alpha_i), \qquad cP = (E_s)_0\sum_i \sinh(\alpha_i),$$

$$\tan[\tfrac{1}{2}\Theta(\alpha)] = \cot[\pi/(\kappa+1)]\tanh(\alpha). \tag{3.1}$$

$(R_s)_0$ and $(E_s)_0 = \hbar c/(R_s)_0$ are the *bare* soliton quantum length and rest energy. In the zero-soliton ground state, all the α_i are real, and the set of quantum numbers N_i ranges over all the integers. Excitation of solitons is achieved by making 'holes' in this distribution of α_i; antisoliton excitations have $\mathrm{Im}(\alpha_i) = \mathrm{i}\pi$, and 'breather' excitonic states, present for $\kappa < 1$, also involve complex α_i. I will describe only the finite soliton density ground state in which the α_i have been removed from a range $-\Lambda < \alpha < \Lambda$.

In the thermodynamic limit, the distribution of real α_i is described by the density $\rho(\alpha_i)=(R_s)_0/(L(\alpha_{i+1}-\alpha_i))$. The zero soliton state density satisfies the linear integral equation

$$2\pi\rho_0(\alpha)=\cosh(\alpha)-\int_{-\infty}^{\infty}\mathrm{d}\alpha'\Theta'(\alpha-\alpha')\rho_0(\alpha'), \tag{3.2}$$

which can be formally solved by a Fourier transform (Bergknoff and Thacker 1979):

$$2\pi\rho_0(\alpha)=\int_{-\infty}^{\infty}\mathrm{d}\alpha'\,(\delta(\alpha-\alpha')-R(\alpha-\alpha'))\cosh(\alpha'), \tag{3.3}$$

$$R(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\;\mathrm{e}^{\mathrm{i}y\alpha}\tilde{K}(y)/(1+\tilde{K}(y)),$$

$$\tilde{K}(y)=\sinh[\pi y(\kappa-1)/(\kappa+1)]/\sinh(\pi y). \tag{3.4}$$

The analogous equation in the presence of a finite soliton density is

$$2\pi\rho(\alpha)-\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha'\,\Theta'(\alpha-\alpha')\rho(\alpha')=\cosh(\alpha)-\int_{-\infty}^{\infty}\mathrm{d}\kappa'\,\Theta'(\alpha-\alpha')\rho(\alpha'), \tag{3.5}$$

$$(N_s/\mathrm{L})\equiv\rho_s=(R_s)_0^{-1}\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha\,\rho(\alpha), \tag{3.6}$$

$$(E_0/L)\equiv\varepsilon_0=(R_s)_0^{-1}(E_s)_0\left(\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha\,\cosh(\alpha)\rho(\alpha)-\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\cosh(\alpha)\Delta\rho(\alpha)\right), \tag{3.7}$$

where ε_0 is the ground state energy per soliton relative to the zero-soliton state, and $\Delta\rho(\alpha)=\rho(\alpha)-\rho_0(\alpha)$. Subtraction of (3.2) from (3.5) gives

$$2\pi\Delta\rho(\alpha)-\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha'\,\Theta'(\alpha-\alpha')\beta(\alpha')=-\int_{-\infty}^{\infty}\mathrm{d}\alpha'\,\Theta'(\alpha-\alpha')\Delta\rho(\alpha'). \tag{3.8}$$

By taking Fourier transforms, this is easily inverted to give

$$\rho(\alpha)=\rho_0(\alpha)+\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha'\,R(\alpha-\alpha')\rho(\alpha'). \tag{3.9}$$

Using this, the expression (3.7) can be manipulated into the form

$$\varepsilon_0=(E_s)_0(R_s)_0^{-1}2\pi\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha\,\rho(\alpha)\rho_0(\alpha), \tag{3.10}$$

where $\rho_0(\alpha)$ is defined by (3.3).

The only problem in the above analysis is that the equation (3.3) for $\rho_0(\alpha)$ is not well defined because it was derived assuming that $\cosh(\alpha)$ had a Fourier transform. The fact that this is *not* so means that some cut-off at large values of α must be imposed. Bergknoff and Thacker (1979) have discussed this point. The clearest way of providing a suitable cut-off is to derive the theory as the limit of a lattice theory, the soluble '*XYZ*' spin chain of Baxter (1972). This, as pointed out by Bergknoff and Thacker, unambiguously identifies the solution of (3.3) as

$$\rho_0(\alpha)=\rho_0(0)\cosh[\tfrac{1}{2}(1+\kappa)\alpha], \tag{3.11}$$

where $\rho_0(0)$ is a divergent multiplicative renormalisation constant.

It is now convenient to obtain a renormalised form of the equations (3.6), (3.9) and (3.10) by normalising $\rho(\alpha)$ by $\rho_0(0)$, and making the scale change $\frac{1}{2}(1+\kappa)\alpha \to \alpha$. Then the final form of the Bethe ansatz equations for finite soliton density is

$$\rho(\alpha)=(2\pi)^{-1}\cosh(\alpha)+\int_{-\Lambda}^{\Lambda} d\alpha'\, R(\alpha-\alpha')\rho(\alpha'), \tag{3.12}$$

$$\rho_s = R_s^{-1}\int_{-\Lambda}^{\Lambda} d\alpha\, \rho(\alpha), \tag{3.13}$$

$$\varepsilon_0 = E_s R_s^{-1}\int_{-\Lambda}^{\Lambda} d\alpha\, \cosh(\alpha)\rho(\alpha), \tag{3.14}$$

$$R(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{i\alpha y}\tilde{\boldsymbol{R}}(y), \tag{3.15}$$

$$\tilde{\boldsymbol{R}}(y)=\tfrac{1}{2}[1-\tanh(\tfrac{1}{2}\pi y)/\tanh(\tfrac{1}{2}\pi\kappa y)]. \tag{3.16}$$

Here E_s and R_s are *renormalised* constants with dimensions of energy and length respectively. The implication in (3.12)–(3.16) that these are precisely the renormalised soliton rest energy and quantum length related by $E_s = \hbar c/R_s$ must be explicitly checked.

The identification of E_s with the soliton rest energy is easily checked by calculating the limiting value of the soliton chemical potential $\mu_s = d\varepsilon_0/d\rho_s$ as $\rho_s \to 0$. For small Λ and α, $\rho(\alpha) \sim 1/2\pi$, and

$$\rho_s \sim R_s^{-1}(\Lambda/\pi), \qquad \varepsilon_0 \sim E_s R_s^{-1}(\Lambda/\pi), \qquad \mu_s \sim E_s, \tag{3.17}$$

so this is verified. The identification of R_s is tested by the limiting value of the *quantum fluctuation parameter* $\exp(\varphi)$ discussed in the last section:

$$e^{2\varphi} = \pi E_s R_s (\rho_s/\mu_s)^{1/2}(d\rho_s/d\mu_s)^{1/2}, \tag{3.18}$$

which defines the quantum soliton length R_s. At high soliton density, when the mass term becomes irrelevant, this must go over into the appropriate value for the mass*less* Thirring model, which is independent of soliton density, and given by

$$e^{2\varphi} \to 2\kappa/(1+\kappa)=\beta^2/4\pi=(1+g/\pi)^{-1}, \qquad \rho_s \to \infty. \tag{3.19}$$

This behaviour will be verified in § 5, and confirms the identification of R_s in (3.13) and (3.14) with the *definition* of the quantum soliton length R_s implicit in (3.18). Incidentally, this identification leads to the result that $\exp(2\varphi) \to 1$ in the low soliton density limit for *all* $\kappa > 0$. This value is in fact always characteristic of a dilute 1D quantum fluid of particles without internal degrees of freedom, whether fermions or bosons.

By following the general discussion in Haldane (1981b), the integral equations for the chemical potential μ_s and quantum fluctuation parameter are easily found by taking derivatives of (3.12) with respect to Λ. Then

$$\tau(\alpha)=R(\alpha-\Lambda)+\int_{-\Lambda}^{\Lambda} d\alpha'\, R(\alpha-\alpha')\tau(\alpha'), \tag{3.20}$$

$$e^{\varphi} = 1+\int_{-\Lambda}^{\Lambda} d\alpha\, \tau(\alpha), \tag{3.21}$$

$$\mu_s = E_s\, e^{-\varphi}\Big(\cosh(\Lambda)+\int_{-\Lambda}^{\Lambda} d\zeta\, \cosh(\alpha)\tau(\alpha)\Big). \tag{3.22}$$

Finally, I note that the integral operator in (3.12) and (3.20) is of a very well behaved type, making these equations eminently suitable for numerical solution by systematic iteration. It is symmetric and definite (positive for $\kappa > 1$ and negative for $\kappa < 1$), and its eigenvalues are real and satisfy the inequality

$$\lambda_n \int_{-\Lambda}^{\Lambda} \mathrm{d}\alpha\, R(\alpha) > 1. \tag{3.23}$$

For $\kappa > \frac{1}{3}$, $|\lambda_n| > 1$ and the iterative solution is absolutely convergent. For $\kappa > \frac{1}{3}$, for sufficiently large Λ, eigenvalues eventually appear in the range $-1 < \lambda_n < 2(1-\kappa^{-1}) < 0$; this means the direct iterative solution is no longer convergent, but the solution can still be obtained numerically by successive iteration from solutions with a nearby value of κ and Λ.

4. Properties of the kernel of the Bethe ansatz integral equation

In this section, I describe some properties of the kernel function $R(\alpha)$ (3.15). The kernel is given in terms of its Fourier transform:

$$R(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}y\, \tilde{R}(y)\, \mathrm{e}^{\mathrm{i}\alpha y}, \qquad \tilde{R}(y) = \tfrac{1}{2}[1 - \tanh(\tfrac{1}{2}\pi y)/\tanh(\tfrac{1}{2}\pi\kappa y)]. \tag{4.1}$$

$R(\alpha)$ diverges as $\kappa \to 0$, but is otherwise finite. $\tilde{R}(y)$ is entire except for poles along the imaginary axis at $y = (2n+1)\mathrm{i}$, and at $y = 2m\mathrm{i}/\kappa$, $m \neq 0$; as $\kappa \to \infty$, these latter poles coalesce into a branch cut. $\tilde{R}(y)$ can also be written in a factorised form suitable for solving certain Wiener–Hopf equations generated by the high soliton density limit, $\Lambda \to \infty$:

$$\tilde{R}(y) = 1 - \Upsilon(\tfrac{1}{2}\mathrm{i}y)\Upsilon(-\tfrac{1}{2}\mathrm{i}y),$$

$$\Upsilon(z) = \left(\frac{\kappa+1}{2\pi\kappa}\right)^{1/2} \Gamma(\tfrac{1}{2}+z)\left(\frac{\exp[-z\kappa\ln(\kappa)]\Gamma(1+\kappa z)}{\exp[-z(\kappa+1)\ln(\kappa+1)]\Gamma[1+(\kappa+1)z]}\right). \tag{4.2}$$

The function $\Upsilon(z)$ is entire and free from zeros except on the negative real axis. $\Upsilon(z) \to 1$ as $|z| \to \infty$, $|\arg(z)| < \pi$. $1/\Upsilon(z)$ has only simple poles at $z = -n/(1+\kappa)$, $n = 1, 2, \ldots$, and is otherwise analytic. $\Upsilon(z) = 1$ when $\kappa = 1$.

$R(\alpha)$ vanishes for $\kappa = 1$; it is positive definite for $\kappa > 1$, and negative definite for $\kappa < 1$. For fixed α, $R(\alpha;\kappa)$ is an increasing function of κ. $R(\alpha)/R(0)$ is even, and decreases monotonically from a single stationary point at $\alpha = 0$; the character of the function is somewhat intermediate between a Gaussian and a Lorentzian; it vanishes as $\alpha \to \infty$.

It does not seem possible to give an explicit analytical expression for $R(0)$ as a function of κ; however, some special limits are:

$$\begin{aligned} R(0) &\sim -\ln(2\,\mathrm{e}^{C}/\pi\kappa)/(\pi^2\kappa) && \kappa \to 0 \\ &= 0 && \kappa = 1 \\ &= 1/2\pi^2 && \kappa = 2 \\ &\sim \ln(2)/\pi^2 - \mathrm{O}(\kappa^{-2}) && \kappa \to \infty. \end{aligned} \tag{4.3}$$

C is Euler's constant.

For large α, $R(\alpha)$ eventually decays exponentially for $\kappa<\infty$, but has algebraic limiting behaviour when $\kappa=\infty$: as $|\alpha|\to\infty$,

$$\begin{aligned} R(\alpha) &\sim \pi^{-1}\tan[\tfrac{1}{2}\pi(\kappa-1)]\exp(-|\alpha|) && \kappa<2\\ &\sim 2\pi^{-2}|\alpha|\exp(-|\alpha|) && \kappa=2\\ &\sim(\pi\kappa)^{-1}\tan(\pi/\kappa)\exp(-2|\alpha|/\kappa) && 2<\kappa<\infty. \end{aligned}\tag{4.4}$$

The change in asymptotic behaviour at $\kappa=2$ is responsible for the MTM cut-off problem discussed by Bergknoff and Thacker (1979): for $\kappa<2$, a simple 'rapidity' cut-off can be used, but for $\kappa>2$, the lattice cut-off of the '*XYZ*' model must be used. For large κ, the asymptotic behaviour separates into two regions:

$$\begin{aligned} R(\alpha) &\sim(1/4\alpha^2) && 1\ll\alpha\ll\kappa\\ &\sim(1/4\alpha^2)(2|\alpha|/\kappa)^2\exp(-2|\alpha|/\kappa) && \kappa\ll\alpha. \end{aligned}\tag{4.5}$$

The change from algebraic to exponential behaviour when $|\alpha|\sim\frac{1}{2}\kappa$ defines a crossover length scale R^* corresponding to the soliton density when $\Lambda\sim\kappa$; this vanishes in the limit $\kappa\to\infty$. As will be seen, $R^*\sim R_s(\kappa)^{-1/2}\exp(-\frac{1}{2}\kappa)$.

In the limit $\kappa=\infty$, $R(y)$ becomes especially simple:

$$R(y)=[1+\exp(\pi|y|)]^{-1},\qquad \Upsilon(z)=(2\pi)^{-1/2}\Gamma(\tfrac{1}{2}+z)\,e^{z}\,e^{-z\ln(z)}.\tag{4.6}$$

$R(\alpha)$ can be expressed as

$$R(\alpha)=(4\pi)^{-1}\int_{-\infty}^{\infty}dt\frac{\operatorname{sech}(t)}{(\frac{1}{2}\pi)^2+(\alpha-t)^2}\qquad \kappa=\infty.\tag{4.7}$$

Apart from a scale factor, this coincides with a similar kernel function $R^{G}(\alpha)\equiv\frac{1}{2}\pi R(\frac{1}{2}\pi\alpha)$ introduced by Griffiths (1964) in connection with the isotropic antiferromagnetic Heisenberg chain. The limit $\kappa\to\infty$ of the SG/MTM in fact describes the critical behaviour of a 1D quantum fluid at its density-wave instability point, and is a very important limit of the model (Haldane 1981d).

Though $R(\alpha)$ diverges as $\kappa\to0$, various limits exist. In particular,

$$\tilde{R}_1(y)=\lim_{\kappa\to0}[-\kappa\tilde{R}(y)]=\tanh(\tfrac{1}{2}\pi y)/\pi y,\tag{4.8}$$

$$R_1(\alpha)=\lim_{\kappa\to0}[-\kappa R(\alpha)]=\pi^{-2}\ln|\coth(\tfrac{1}{2}\alpha)|.\tag{4.9}$$

This explicit form for the rescaled kernel $R_1(\alpha)$ exhibits the development of a logarithmic singularity at $\alpha=0$. A second limit probes the small-α region:

$$\begin{aligned} &R(\alpha)\sim R(0)-\kappa^{-1}R_2(\alpha/\kappa), \qquad |\alpha|\ll1,\ \kappa\to0,\\ &R_2(u)=\lim_{\kappa\to0}[-\kappa(R(0)-R(\kappa u))]=\pi^{-1}\int_0^{\infty}dx\frac{2\sin^2(\frac{1}{2}xu)}{\exp(\pi x)-1} \end{aligned}\tag{4.10}$$

which can be written as

$$R_2(u)=\tfrac{1}{2}\pi^{-2}[\psi(1+iu/\pi)+\psi(1-iu/\pi)-2\psi(1)]$$

where $\psi(x)$ is Euler's psi function. For large $|u|$, $R_2(u)\sim\pi^{-2}\ln|u\,e^{C}/\pi|$; when combined with the limiting behaviour of $R(0)$ given by (4.3), $[\kappa R(0)-R_2(\alpha/\kappa)]$ with $|\alpha|/\kappa$ *large* matches onto the *small-*α form of $R_1(\alpha)$.

5. The low and high soliton density limits ($\kappa > 0$)

In this section, I examine the properties of the soliton fluid ground state in the limits of low and high density, including the crossover in high-density behaviour as $\kappa \to \infty$. The behaviour in the classical limit $\kappa \to 0$ is more complex, and is dealt with in a separate section.

When κ is non-zero, the kernel function $R(\alpha)$ is non-singular, and the properties of the system in the dilute limit are analytic in Λ (and hence in $|\rho_s R_s|$). Power series in α and Λ can be developed for the solutions $\rho(\alpha)$ and $\tau(\alpha)$ of (3.12) and (3.20):

$$2\pi\rho(\alpha) = 1 + 2R(0) + O(\Lambda^2) + O(\alpha^2),$$
$$\tau(\alpha) = R(0)[1 + 2R(0) + O(\Lambda^3)] + O(\alpha\Lambda). \tag{5.1}$$

Substitution into (3.13), (3.14), (3.20) and (3.21), together with relations (2.13) and (2.15), easily leads to the expansions

$$\varepsilon_0 = E_s\rho_s[1 + \tfrac{1}{6}(\pi\rho_s R_s)^2 - \tfrac{2}{3}R(0)(\pi\rho_s R_s)^3 + O(\rho_s R_s)^4], \tag{5.2}$$
$$\mu_s = E_s[1 + \tfrac{1}{2}(\pi\rho_s R_s)^2 - \tfrac{8}{3}R(0)(\pi\rho_s R_s)^3 + O(\rho_s R_s)^4], \tag{5.3}$$
$$e^{\varphi} = [1 + 2R(0)(\pi\rho_s R_s) + O(\rho_s R_s)^2] \tag{5.4}$$
$$v_N = c(\pi\rho_s R_s)[1 - 8R(0)(\pi\rho_s R_s) + O(\rho_s R_s)^2], \tag{5.5}$$
$$v_S = c(\pi\rho_s R_s)[1 - 4R(0)(\pi\rho_s R_s) + O(\rho_s R_s)^2], \tag{5.6}$$
$$v_J = c(\pi\rho_s R_s)[1 - \tfrac{1}{2}(\pi\rho_s R_s)^2 + \tfrac{8}{3}R(0)(\pi\rho_s R_s)^3 + O(\rho_s R_s)^4]. \tag{5.7}$$

The lowest-order coupling-dependent terms are controlled by $R(0)$. This diverges to $-\infty$ as $\kappa \to 0$, and the radius of convergence of the series (5.2)–(5.7) then shrinks to zero. In the low-density limit, $\exp(\varphi) \to 1$, as the dimensionless coupling constant of the contact force between solitons diverges, while matrix elements for gradient terms in the interactions vanish. As the contact coupling diverges, the properties of the system approach those of a free spinless Fermi gas, which can always be assumed to have an infinite contact coupling, as pointed out in Girardeau's (1960) study of hard-core systems in one space dimension. While the lowest-order corrections to v_N and v_S are coupling dependent, those to v_J are not; this is a feature due to the Galilean invariance that appears in the low-density limit.

The properties in the high-density limit as $\Lambda \to \infty$ involve the Wiener–Hopf problem

$$g_\infty(\alpha) = f(\alpha) + \int_0^\infty d\alpha' R(\alpha - \alpha') g_\infty(\alpha') \qquad \alpha > 0 \tag{5.8}$$

which has the solution

$$g_\infty(\alpha) = f(\alpha) + \int_0^\infty d\alpha' \, [S_0(\alpha, \alpha') + S_1(\alpha - \alpha')] f(\alpha),$$
$$S_0(\alpha, \alpha') = \frac{-i}{(2\pi)^2} \int_{-\infty}^\infty dy \int_{-\infty}^\infty dz \frac{\exp[i(\alpha y - \alpha' z)]}{(y - z + i0^+)} [Y(\tfrac{1}{2}iy)Y(-\tfrac{1}{2}iz)]^{-1},$$
$$S_1(\alpha) = \frac{1}{2\pi} \int_{-\infty}^\infty dy \, e^{iy\alpha} \frac{\tilde{R}(y)}{1 - \tilde{R}(y)}, \tag{5.9}$$

where $\tilde{R}(y) = 1 - \Upsilon(\frac{1}{2}\mathrm{i}y)\Upsilon(-\frac{1}{2}\mathrm{i}y)$ is the factorisation (4.2). S_0 can be expanded as

$$S_0(\alpha, \alpha') = 2(\kappa+1) \sum_{m,n=1}^{\infty} \frac{a_m(\kappa)a_n(\kappa)}{(m+n)} \exp\left(\frac{-2m\alpha}{(\kappa+1)}\right) \exp\left(\frac{-2n\alpha'}{(\kappa+1)}\right), \tag{5.10}$$

where $a_n(\kappa)$ are the residues of the simple poles of $1/\Upsilon(z)$ at $z = -n/(\kappa+1)$:

$$\frac{1}{\Upsilon(z)} = 1 + \sum_{n=1}^{\infty} \frac{a_n(\kappa)}{[z + n/(\kappa+1)]}. \tag{5.11}$$

These results can be used to invert the integral equation

$$g(\alpha) = f(\alpha) + \int_0^{\Lambda} \mathrm{d}\alpha' \, R(\alpha - \alpha') f(\alpha'), \tag{5.12}$$

which becomes

$$g(\alpha) = g_\infty(\alpha) + \int_{\Lambda}^{\infty} \mathrm{d}\alpha' \, [S_0(\alpha, \alpha') + T(\alpha - \alpha')] g(\alpha'), \tag{5.13}$$

$$T(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{e}^{-\mathrm{i}\alpha y} \tilde{T}(y),$$

$$\tilde{T}(y) = \frac{(\tilde{R}(y))^2}{1 - \tilde{R}(y)} = \frac{\frac{1}{2}\{\sinh[\frac{1}{2}\pi(\kappa-1)y]\}^2}{\cosh(\frac{1}{2}\pi y)\sinh(\frac{1}{2}\pi\kappa y)\sinh[\frac{1}{2}\pi(\kappa+1)y]}. \tag{5.14}$$

The Bethe ansatz integral equations can be manipulated into the form (5.12) for studying the high-density limit:

$$\bar{\rho}(\alpha) = \mathrm{e}^{-\alpha} + \int_0^{\bar{\Lambda}} \mathrm{d}\alpha' \, R(\alpha - \alpha') \bar{\rho}(\alpha'), \tag{5.15}$$

$$\bar{\tau}(\alpha) = R(\alpha) + \int_0^{\bar{\Lambda}} \mathrm{d}\alpha' \, R(\alpha - \alpha') \bar{\tau}(\alpha'), \tag{5.16}$$

where $\bar{\Lambda} = 2\Lambda$, and the physical properties are given by

$$\rho_s = R_s^{-1} \frac{\mathrm{e}^{\bar{\Lambda}/2}}{2\pi} \int_0^{\bar{\Lambda}} \mathrm{d}\alpha \, \bar{\rho}(\alpha), \qquad \varepsilon_0 = E_s R_s^{-1} \frac{\mathrm{e}^{\bar{\Lambda}}}{4\pi} \int_O^{\bar{\Lambda}} \mathrm{d}\alpha \, (\mathrm{e}^{-\alpha} + \mathrm{e}^{(\bar{\alpha}-\bar{\Lambda})}) \bar{\rho}(\alpha),$$

$$\mathrm{e}^{\varphi} = \left(1 + \int_0^{\bar{\Lambda}} \mathrm{d}\alpha \, \bar{\tau}(\alpha)\right),$$

$$\mu_s = \tfrac{1}{2} E_s \, \mathrm{e}^{-\varphi} \mathrm{e}^{\bar{\Lambda}/2} \left(1 + \mathrm{e}^{-\bar{\Lambda}} + \int_0^{\bar{\Lambda}} \mathrm{d}\alpha \, (\mathrm{e}^{-\alpha} + \mathrm{e}^{(\alpha-\bar{\Lambda})}) \bar{\tau}(\alpha)\right). \tag{5.17}$$

When $\bar{\Lambda} \to \infty$, the solutions of (5.15) and (5.16) are

$$\bar{\rho}_\infty(\alpha) = (\Upsilon(\tfrac{1}{2}))^{-1} \left(-\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{x + \mathrm{i}} (\Upsilon(-\tfrac{1}{2}\mathrm{i}x))^{-1}\right), \tag{5.18}$$

$$\bar{\tau}_\infty(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\alpha x} [(\Upsilon(-\tfrac{1}{2}\mathrm{i}x))^{-1} - 1]. \tag{5.19}$$

These can be expanded as

$$\bar{\rho}_\infty(\alpha) = (\Upsilon(\tfrac{1}{2}))^{-1} \sum_{n=1}^{\infty} a_n(\kappa) \frac{\exp[-2n\alpha/(\kappa+1)]}{[\frac{1}{2} - n/(\kappa+1)]}, \tag{5.20}$$

$$\bar{\tau}_\infty(\alpha)=2\sum_{n=1}^{\infty} a_n(\kappa)\exp[-2n\alpha/(\kappa+1)]. \tag{5.21}$$

(Note that $1/\Upsilon(-\frac{1}{2})=0$ for $\kappa\neq 1$.) For $\kappa<\infty$, these expressions vanish exponentially as $\alpha\to\infty$. Note also that the change in asymptotic behaviour of $R(\alpha)$ as $\alpha\to\infty$ that occurs at $\kappa=2$ is *not* reflected in the solution $\rho_\infty(\alpha)$. It was this change that caused the cut-off problem in the MTM for $\kappa>2$, as discussed by Bergknoff and Thacker. The renormalisation has completely eliminated it.

The limiting behaviour of the physical quantities as $\Lambda\to\infty$ is that of the mass*less* Thirring model: substitution of $\bar{\rho}_\infty$ and $\bar{\tau}_\infty$ into (5.17) gives

$$e^{\bar{\Lambda}/2}\sim 2\pi\rho_s R_s\Upsilon(0)\Upsilon(\tfrac{1}{2}), \tag{5.22}$$

$$e^{\varphi}\to e^{\varphi_0}=1/\Upsilon(0)=[2\kappa/(\kappa+1)]^{1/2},$$

$$\varepsilon_0\sim(\pi\hbar c)\tfrac{1}{2}\,e^{-2\varphi_0}\rho_s^2, \qquad \mu_s\sim(\pi\hbar c)\,e^{-2\varphi_0}\rho_s,$$

$$v_S\to c, \qquad v_N\to c\,e^{-2\varphi_0}, \qquad v_J\to c\,e^{2\varphi_0}. \tag{5.23}$$

As noted in § 3, the correct limiting behaviour of $\exp(\varphi)$ confirms the identification of R_s with the renormalised quantum soliton length.

The leading corrections as $\Lambda\to\infty$ can be obtained by iterating equations of the form of (5.13). It is useful to note that for $\alpha>0$, $T(\alpha)$ has an expansion of the form

$$T(\alpha)=\sum_{n=1}^{\infty}\{t_{1n}(\kappa)\exp[-(2n-1)\alpha]+t_{2n}(\kappa)\exp(-2n\alpha/\kappa)+t_{3n}(\kappa)\exp[-2n\alpha/(\kappa+1)]\}. \tag{5.24}$$

Thus expansions for the physical properties given by (5.17) will involve sums of terms of the form $\exp\{-[n_1\bar{\Lambda}+n_2(2/\kappa)\bar{\Lambda}+n_3(2/\kappa+1)\bar{\Lambda}]\}$. It is useful to introduce the solutions of the associated Wiener–Hopf problem:

$$f_n(\alpha,\kappa)=\int_0^{\infty}d\alpha'\;T(\alpha-\alpha')\{f_n(\alpha',\kappa)+\exp[-2n\alpha'/(\kappa+1)]\}. \tag{5.25}$$

These occur in the solution of (5.13) by iteration, but will not be explicitly constructed here.

When $\kappa<\infty$, the leading corrections at large but finite soliton density are controlled by an exponent ε:

$$\varepsilon=4/(\kappa+1)=2(2-e^{2\varphi_0}), \qquad \varepsilon<2 \text{ for } \kappa>1, \qquad 2<\varepsilon<4 \text{ for } \kappa<1. \tag{5.26}$$

The corrections involve a positive coefficient $A(\kappa)$:

$$A(\kappa)=2\pi^{1/2}\frac{\Gamma[\frac{1}{2}+1/(\kappa+1)]}{\Gamma[1/(\kappa+1)]}\left(\frac{\sin[\frac{1}{4}\pi(2-\varepsilon)]}{\frac{1}{4}\pi(2-\varepsilon)}\right)(\kappa)^{\kappa/(\kappa+1)}[2\Upsilon(0)\Upsilon(\tfrac{1}{2})]^{-\varepsilon}$$
$$\times\left(\tfrac{1}{2}(\kappa+1)-\int_{-\infty}^{0}d\alpha\,f_1(\alpha,\kappa)\right). \tag{5.27}$$

When $\kappa=1$, $\varepsilon=2$ and $A(1)=\frac{1}{2}$. The limiting behaviour of the physical quantities is

$$\varepsilon_0=(\pi\hbar c)\{\tfrac{1}{2}\,e^{-2\varphi_0}\rho_s^2[1+2A(\kappa)(2-\varepsilon)^{-1}(\pi\rho_s R_s)^{-\varepsilon}\ldots]+\pi\tan(\tfrac{1}{2}\pi\kappa)(2\pi R_s)^{-2}\},$$

$$\mu_s=(\pi\hbar c)\,e^{-2\varphi_0}\rho_s[1+A(\kappa)(\pi\rho_s R_s)^{-\varepsilon}\ldots],$$

$$e^{\varphi}=e^{\varphi_0}[1-\tfrac{1}{4}A(\kappa)(2-\varepsilon)(\pi\rho_s R_s)^{-\varepsilon}\ldots], \qquad v_S=c[1-\tfrac{1}{2}A(\kappa)\varepsilon(\pi\rho_s R_s)^{-\varepsilon}\ldots],$$

$$v_N=c\,e^{-2\varphi_0}[1+A(\kappa)(1-\varepsilon)(\pi\rho_s R_s)^{\varepsilon}\ldots], \qquad v_J=c\,e^{2\varphi_0}[1-A(\kappa)(\pi\rho_s R_s)^{\varepsilon}\ldots]. \tag{5.28}$$

Various features of (5.28) are of note. The corrections to the ground-state energy diverge as $\rho_s \to \infty$ when $\kappa \geqslant 1$, and the constant term in ε_0 is merely a non-leading correction; however, for $\kappa < 1$, the corrections converge, and the constant energy density $(\pi\hbar c)[\pi \tan(\frac{1}{2}\pi\kappa)](2\pi R_s)^{-2}$ represents the lowering of the energy of the zero-soliton state associated with the opening of the soliton gap. This energy diverges as $\kappa \to 1^-$. When $\kappa = 1$, $\varepsilon = 2$, and the free fermion results of the non-interacting MTM are recovered; there is no renormalisation of $\exp(\varphi)$, which remains equal to 1. When $\varepsilon = 1$ $(\kappa = 3)$, the corrections to v_N change sign; for $\kappa < 3$, the corrections to the chemical potential vanish in the high-density limit, but they diverge when $\kappa > 3$. The leading correction to v_S vanishes as $\varepsilon \to 0$ $(\kappa \to \infty)$.

Finally, I examine the asymptotic high-density behaviour in the regime of *algebraic* decay of the kernel $(1 \ll \Lambda \ll \kappa)$ that takes over as $\kappa \to \infty$. For κ and $\Lambda \gg 1$,

$$
\begin{aligned}
&\Upsilon(\tfrac{1}{2})\bar{\rho}_\infty(\alpha) \sim \bar{\tau}_\infty(\alpha) && 1 \ll \alpha, \\
\bar{\tau}_\infty(\alpha) &\sim 2\sqrt{2}(1/4\alpha^2)\{1 + \alpha^{-1}[\ln(\tfrac{1}{2}\alpha) - \tfrac{5}{6}]\ldots\} && 1 \ll \alpha \ll \kappa \\
&\sim 2\sqrt{2}(1/4\alpha^2)[2\alpha/(\kappa+1)]^2 \exp[-2\alpha/(\kappa+1)] && \kappa \ll \alpha.
\end{aligned}
\tag{5.29}
$$

Substitution of these results into (5.17) gives the leading corrections when $\kappa = \infty$:

$$
\begin{aligned}
\varepsilon_0 &= (\pi\hbar c)\tfrac{1}{4}\rho_s^2\{1 + \tfrac{1}{2}[\ln(\pi\rho_s R_s)]^{-1}\ldots\}, & \mu_s &= (\pi\hbar c)\tfrac{1}{2}\rho_s\{1 + \tfrac{1}{2}[\ln(\pi\rho_s R_s)]^{-1}\ldots\}, \\
e^\varphi &= \sqrt{2}\{1 - \tfrac{1}{4}[\ln(\pi\rho_s R_s)]^{-1}\ldots\}, & v_S &= c\{1 - \tfrac{1}{4}[\ln(\pi\rho_s R_s)]^{-2}\ldots\}, \\
v_N &= \tfrac{1}{2}c\{1 + \tfrac{1}{2}[\ln(\pi\rho_s R_s)]^{-1}\ldots\}, & v_J &= 2c\{1 - \tfrac{1}{2}[\ln(\pi\rho_s R_s)]^{-1}\ldots\}.
\end{aligned}
\tag{5.30}
$$

The leading correction to v_S is one order below that of the other terms. The crossover between these results and the results (5.28) for large but finite κ takes place when $\bar{\Lambda} \sim \kappa$; a precise definition of a crossover length scale R^* can be obtained by equating the value of $\exp(\varphi)$ given by (5.30) when $\pi\rho_s R^* = 1$ to the limiting finite-κ value $\exp(\varphi_0) = [2\kappa/(\kappa+1)]^{1/2}$. From (5.30), $R^* \sim R_s \exp(-\frac{1}{2}\kappa)$; to determine the behaviour of R^* as $\kappa \to \infty$ more precisely, it is necessary to examine the next order of corrections to (5.30). The equation for $\bar{\tau}(\alpha)$ can be written as

$$
\begin{aligned}
[\bar{\tau}(\alpha) - \bar{\tau}_\infty(\alpha)] &= -\int_{\bar{\Lambda}}^{\infty} d\alpha'\, R(\alpha-\alpha')\bar{\tau}_\infty(\alpha') + \int_0^{\bar{\Lambda}} d\alpha'\, R(\alpha-\alpha')[\bar{\tau}(\alpha') - \bar{\tau}_\infty(\alpha')], \\
&\quad -\int_{\bar{\Lambda}}^{\infty} d\alpha'\, R(\alpha-\alpha')\bar{\tau}_\infty(\alpha') \sim -(\sqrt{2}/2\bar{\Lambda}^2)\int_{\bar{\Lambda}-\alpha}^{\infty} d\alpha'\, R(\alpha').
\end{aligned}
\tag{5.31}
$$

In the limit $\Lambda \to \infty$, the derivative of this equation can be related to the equation for $\bar{\tau}_\infty(\alpha)$ itself, leading to the result

$$
\bar{\tau}(\alpha) \sim \bar{\tau}_\infty(\alpha) - \left(\frac{1}{2\bar{\Lambda}^2}\right)\int_{\bar{\Lambda}-\alpha}^{\infty} d\alpha'\, \bar{\tau}_\infty(\alpha') + O(\bar{\Lambda}^{-3}).
\tag{5.32}
$$

Substitution into the expression for $\exp(\varphi)$ gives the expansion

$$
e^\varphi = \sqrt{2}\{1 - \tfrac{1}{2}\bar{\Lambda}^{-1} - \tfrac{1}{2}\bar{\Lambda}^{-2}[\ln(\bar{\Lambda}) - 2\ln(A)] + O(\bar{\Lambda}^{-3})\},
\tag{5.33}
$$

which (when taken with (5.22) which has no corrections to this order) leads to the result

$$
R^* \sim A R_s(\tfrac{1}{2}\pi\kappa)^{-1/2}\exp(-\tfrac{1}{2}\kappa) \qquad \text{as } \kappa \to \infty.
\tag{5.34}
$$

The constant A is given by

$$-4\ln(A) = C - \tfrac{5}{6} + \lim_{\varepsilon\to 0^+}\left\{\ln(\varepsilon) + \int_\varepsilon^\infty \frac{dy}{y}\left(\frac{\sin(2\pi y)}{2\pi y}\right)\left(\frac{\Gamma(\frac{1}{2}+y)}{\Gamma(\frac{1}{2})}\right) e^y\, e^{-y\ln(y)}\right]. \tag{5.35}$$

The limiting behaviour in the high-density limit obtained in this section can also be obtained by the scaling theory or renormalisation group treatment of the MTM mass perturbation expansion, apart from the values of numerical constants.

6. The crossover to the classical limit as $\kappa \to 0$ ($\beta^2 \to 0$)

In this section, I will make contact between the Bethe ansatz solution of the QSG, and the ground-state properties of the classical ($\beta^2 = 0$) sine-Gordon system

$$H = \tfrac{1}{8}E_s \int_0^L dx\left\{\tfrac{1}{2}R_b[\nabla\theta(x)]^2 + R_b^{-1}[1-\cos(\theta(x))]\right\},$$

$$\theta(x+L) = \theta(x) + 2\pi N_s. \tag{6.1}$$

The $N_s = 1$ (single-soliton) ground state has broken translational symmetry, and in the limit $L \to \infty$ is given by

$$\theta(x) = 4\tan^{-1}\{\exp[(x-x_0)/R_b]\}, \qquad E_0 = E_s. \tag{6.2}$$

Frank and van der Merwe (FdvM) (1949) obtained the finite soliton density ground state: it is a regular evenly spaced lattice of solitons (localised regions where the phase $\theta(x)$ slips by 2π), with ground-state energy density and soliton density given parametrically in terms of the modulus k of complete elliptic functions of the first and second kinds, $K(k)$ and $E(k)$, with complementary modulus $k' = (1-k^2)^{1/2}$.

$$\rho_s R_b = 1/(2kK), \qquad \varepsilon_0 = E_s\rho_s(E - \tfrac{1}{2}k'^2K)/k. \tag{6.3}$$

In the high-density limit ($k \to 0$)

$$\varepsilon_0 = \tfrac{1}{4}\pi^2 E_s R_b \rho_s^2[1 + \tfrac{1}{32}(\pi\rho_s R_b)^{-4}\ldots] + \tfrac{1}{8}E_s R_b^{-1}. \tag{6.4a}$$

This should be compared with the limit of (5.28) as $\kappa \to 0$:

$$\varepsilon_0 = \tfrac{1}{4}\pi^2 E_s(R_s/\pi\kappa)\rho_s^2[1 + O(\rho_s R_s)^{-4}\ldots] + \tfrac{1}{8}E_s(R_s/\pi\kappa)^{-1}. \tag{6.4b}$$

The comparison shows that as $\kappa \to 0$, the classical soliton width R_b of (6.1) is related to the quantum soliton width by $R_s \sim \pi\kappa R_b$: the classical soliton width is thus precisely given by the principal quantum *breather* length $R_b = R_s/[2\sin(\frac{1}{2}\pi\kappa)]$ as $\kappa \to 0$. An interpretation of this is that as $\kappa \to 0$, the vacuum becomes 'softer' against virtual breather fluctuations, and the 'bare' soliton becomes 'dressed' by a cloud of virtual breathers, so that its effective size in the classical limit is controlled by the breather length scale.

The long-range order of the periodic soliton lattice corresponds to '$2k_F$' periodicity of the density correlations, with vanishing quantum fluctuation parameter $\exp(\varphi)$. Equation (2.14) shows that $\exp(2\varphi) \propto E_s R_s = \pi\kappa E_s R_b$ vanishes linearly with κ as the limit $\kappa \to 0$ is taken at finite soliton density.

In the *dilute* classical soliton limit, $\rho R_b \ll 1$ ($k' \to 0$), the FvdM solution behaves as

$$\varepsilon_0 = E_s\rho_s[\![1 + 4\exp[-1/(\rho_s R_b)] + O\{\exp[-1/(\rho_s R_b)]\}^2]\!]. \tag{6.5}$$

In this limit, the solitons are almost free, except for an exponentially weak repulsive effective interaction $4E_s\exp(-|r|/R_b)$ between solitons, which has the same fall-off as the strain field associated with the single-soliton solution (6.2): $\nabla\theta(x) = 2R_b^{-1}\,\mathrm{sech}[(x-x_0)/R_b]$.

As noted in § 4, the kernel $R(\alpha)$ of the Bethe ansatz equation (3.12) diverges as $\kappa\to 0$, but the function $R_1(\alpha) = -\kappa R(\alpha)$ (4.8) has the limit $\pi^{-2}\ln|\coth(\frac{1}{2}\alpha)|$. When the pseudomomenta density function $\rho(\alpha)$ is renormalised by a factor $(\pi\kappa)^{-1}$, equations (3.12)–(3.14) have the following limit as $\kappa\to 0$:

$$\tfrac{1}{2}\cosh(\alpha) = \int_{-\Lambda}^{\Lambda} d\alpha'\,\ln|\coth[\tfrac{1}{2}(\alpha-\alpha')]|\rho_1(\alpha') \tag{6.6}$$

$$\rho_s = R_b^{-1}\int_{-\Lambda}^{\Lambda} d\alpha\,\rho_1(\alpha), \tag{6.7}$$

$$\varepsilon_0 = E_s R_b^{-1}\int_{-\Lambda}^{\Lambda} d\alpha\,\cosh(\alpha)\,\rho_1(\alpha). \tag{6.8}$$

The derivative of the integral equation (6.6) gives

$$\tfrac{1}{2}\sinh(\alpha) = \int_{-\Lambda}^{\Lambda} d\alpha'\,[\sinh(\alpha-\alpha')]^{-1}\rho_1(\alpha). \tag{6.9}$$

This singular integral equation is put into Cauchy form by the change of variable $t = \tanh(\alpha)$; it is then easily solved to give

$$2\pi\rho_1(\alpha) = \mathrm{sech}(\Lambda)[\tanh^2(\Lambda)-\tanh^2(\alpha)]^{-1/2}[C(\Lambda)\,\mathrm{sech}(\alpha)+\cosh(\alpha)]. \tag{6.10}$$

$C(\Lambda)$ is an undetermined constant that must be found by substitution back into the original equation (6.6).

Unfortunately, I was not able to find $C(\Lambda)$ explicitly for general Λ, nor carry out the integrals (6.7), (6.8) that should lead to the FvdM result (6.3), despite the obvious connection of integrals of $\rho_1(\alpha)$ with elliptic integrals. However, for small Λ, evaluation of $C(\Lambda)$ is straightforward as the hyperbolic functions can be linearised, with the result

$$2\pi\rho_1(\alpha) \sim [\ln(4/\Lambda)]^{-1}(\Lambda^2-\alpha^2)^{-1/2} + O(\Lambda) \qquad \Lambda \ll 1. \tag{6.11}$$

Substitution of (6.11) into (6.7) and (6.8) leads to the low-density FvdM result (6.5), with $\Lambda \sim k'$ as $\Lambda \to 0$.

It is now of interest to see how the QSG results behave for both ρ_s and κ small but finite: there is an obvious discrepancy between the two limits of the quantum fluctuation parameter $\exp(\varphi)\to 0$ for $\kappa\to 0$, ρ_s finite, and $\exp(\varphi)\to 1$ for $\rho_s\to 0$, κ finite. For small α and κ, $R(\alpha)$ has the form (4.10):

$$R(\alpha) \sim R(0) - \kappa^{-1}R_2(\alpha/\kappa), \qquad R(0) = -\ln(2e^C/\pi\kappa)/(\pi^2\kappa),$$
$$R_2(u) \sim \pi^{-2}\zeta(3)(u/\pi)^2 \quad u\to 0, \qquad R_2(u) \sim \pi^{-2}\ln|e^C u/\pi| \quad u\to\infty \tag{6.12}$$

($\zeta(3) = 1.20205\ldots$ is the Riemann zeta function). For $|\alpha| \ll \kappa$, $R(\alpha)\sim R(0)$, constant, while for $|\alpha| \gg \kappa$, the $\kappa = 0$ form $\sim\ln|\frac{1}{2}\alpha|/(\pi^2\kappa)$ is recovered. For $1 \gg \Lambda \gg \kappa$, to a first approximation, the FvdM result (6.5) can be inserted into (2.14) to give the limiting behaviour of $\exp(\varphi)$ as $\kappa\to 0$:

$$e^{\varphi} \sim \pi(\tfrac{1}{2}\kappa)^{1/2}(\rho_s R_b)\exp(1/4\rho_s R_b), \qquad \kappa \ll \Lambda \sim 4\exp(-1/2\rho_s R_b) \ll 1. \tag{6.13}$$

This result reflects the physics of small-amplitude, zero-point fluctuations of the solitons in the weak potential well due to the exponential force between neighbouring solitons. As the soliton density decreases, this potential well becomes weaker, and $\exp(\varphi)$, which is essentially the ratio of the amplitude of zero-point fluctuations to the mean soliton separation, increases. When the semiclassical result fails, $\exp(\varphi)\sim(\rho_s R_b)\sim 1/\ln(1/\kappa)\ll 1$; the soliton is moving in an effective potential of form $\sim\cosh(r/R_b)$, and this can be seen to be precisely the condition that the harmonic approximation for zero-point motion fails.

In the other limit $\Lambda\ll\kappa$, the Bethe ansatz equations with constant kernel $R(\alpha)\sim R(0)$ are easily solved, leading to the result

$$e^{\varphi}\sim(1+2\pi R(0)\rho_s R_s), \qquad \Lambda\sim\pi\rho_s R_s/(1+2\pi R(0)\rho_s R_s). \tag{6.14}$$

Since $R(0)$ is negative, $\exp(\varphi)$ given by (6.14) apparently vanishes at a critical density given by $\rho_s\bar{R}=1$, where

$$\bar{R}=2R_b\ln(2e^{C}/\pi\kappa). \tag{6.15}$$

Actually, this is spurious, since the condition $\Lambda<\kappa$ means that (6.14) is only valid when $\exp(\varphi)>\pi^2\rho_s R_b$, which is the same crossover condition as that obtained from the opposite FvdM limit. As $\kappa\to 0$ the crossover region between (6.13) and (6.14) becomes sharper and sharper, and $\bar{R}$ defined by (6.15) is a well defined characteristic length. The crossover region is given by

$$|\rho_s\bar{R}-1|\lesssim\pi^2R_b/\bar{R}=\tfrac{1}{2}\pi^2/\ln(2\,e^{C}/\pi\kappa). \tag{6.16}$$

As $\kappa\to 0$, a universal crossover function between these two regions can be defined: for $|\rho_s\bar{R}-1|\ll 1$, as $\kappa\to 0$,

$$e^{\varphi}=(R_b/\bar{R})F[(\bar{R}/R_b)(1-\rho_s\bar{R})],$$

$$F(x)\sim x \qquad x\gg 1; \qquad F(x)\sim(e^{C}/\pi)^{1/2}\exp(\tfrac{1}{4}x) \qquad (x\ll -1). \tag{6.17}$$

The breakdown of the solution (6.14) can be studied by including the limiting quadratic behaviour of R_2 in the kernel $R(\alpha)$. The solution to lowest order in this correction is easily developed:

$$e^{\varphi}=(1-\rho_s\bar{R})[1+(2\pi^2\zeta(3)/3)(\bar{R}/R_b)^{-3}(1-\rho_s\bar{R})^{-3}\ldots]. \tag{6.18}$$

This implies the asymptotic behaviour

$$F(x)\sim x+(2\pi^2\zeta(3)/3)/x^2\ldots \qquad \text{as } x\to\infty. \tag{6.19}$$

7. Discussion

This paper has presented the *renormalised* Bethe ansatz equations describing the quantum fluid ground state of the sine-Gordon/massive Thirring model, obtained from the unrenormalised equations derived by Bergknoff and Thacker (1979). The 'quantum fluctuation parameter' $\exp(\varphi)$ that controls the quantum fluid correlation functions was extracted from the Bethe ansatz equations, and used to characterise the basic low-energy physics which is described in terms of a recent theory of 1D quantum fluids (or 'Luttinger liquids', Haldane (1980, 1981a)).

The properties of the soliton in both high- and low-density limits were explicitly given, as well as in the semiclassical ($\beta^2\to 0$) and critical ($\beta^2\to 8\pi$) limits of the SG coupling parameter; various crossovers were described in detail. A number of length scales appear in the problem. The fundamental length is the quantum soliton length $R_s = \hbar/m_s c$, where m_s is the soliton mass. In the limit $\beta^2 \to 0$, the quantum *breather* length $R_b = \hbar/m_b c \gg R_s$ becomes the characteristic length of the theory: it is this length that characterises the width of the classical soliton. A second length $\bar{R} \gg R_b$ controls the crossover between the dilute quantum fluid behaviour and the semiclassical region of small zero-point fluctuations about the classical soliton lattice state described by Frank and van der Merwe (1949). Similarly, a length $R^* \ll R_s$ controls critical behaviour in the high-density limit as $\beta^2 \to 8\pi$.

In the region $\beta^2 < 4\pi$, the ground-state energy shift associated with the opening of the soliton gap is finite: this energy density was obtained, and is given by $\frac{1}{4}E_s R_s^{-1} \tan(\frac{1}{2}\pi\kappa)$, where $\kappa = (\beta^2/8\pi)/(1-\beta^2/8\pi)$, and E_s is the soliton rest energy.

Expansions for the various ground-state and low-energy excitation spectrum properties were given in the various limits. If curves describing their behaviour in intermediate regions are required, they can be obtained from the integral equations presented here, which are easily solved numerically by iteration.

A number of applications of the SG results can be suggested. The limiting critical behaviour as $\beta^2 \to 8\pi$ is closely connected with critical behaviour at the density-wave instability of 1D quantum fluids (Haldane 1981d), such as seen close to the isotropic point of 1D antiferromagnets (Haldane 1980). As stressed recently by Black and Emery (1981), this behaviour is directly related to critical behaviour associated with the Kosterlitz–Thouless transition in various 2D classical models, such as the *XY* model and Potts model variants. A direct translation of the results presented here into 2D classical language can be made, with the quantum fluctuation parameter now becoming the parameter controlling the power-law correlations of the classical model. From the integral equations presented here, the full crossover functions are exactly calculable.

A second application to 2D classical problems involves the crossover behaviour as $\beta^2 \to 0$. Here the QSG becomes relevant to critical behaviour at the commensurate–incommensurate transition of 2D adsorbed monolayers that become incommensurate in one direction, with domain walls playing an analogous role to 1D solitons (see e.g. Schulz (1980)). A detailed report including the translation of the results of § 6 into the language of this problem has already been prepared (Haldane and Villain 1981).

References

Baxter R J 1972 *Ann. Phys., NY* **70** 193
Bergknoff H and Thacker H 1979 *Phys. Rev.* D **19** 3666
Black J L and Emery V J 1981 *Phys. Rev.* B **23** 429
Coleman S 1975 *Phys. Rev.* D **11** 2088
Frank F C and van der Merwe J H 1949 *Proc. R. Soc.* A **198** 205
Girardeau M 1960 *J. Math. Phys.* **1** 516
Griffiths R B 1964 *Phys. Rev.* **133** A 768
Gupta N and Sutherland B 1976 *Phys. Rev.* A **14** 1790
Haldane F D M 1980 *Phys. Rev. Lett.* **45** 1358
—— 1981a *J. Phys. C: Solid State Phys.* **14** 2585
—— 1981b *Phys. Lett* **81A** 153
—— 1981c *ILL Preprint SP-80/146* to be published
—— 1981d *J. Physique* to be published
—— 1982 *J. Phys. C: Solid State. Phys.* **15** to be published
Haldane F D M and Villain J 1981 to be published

Lieb E H 1963 *Phys. Rev.* **130** 1616
Overhauser A W 1965 *Physics* **1** 307
Schulz H J 1980 *Phys. Rev.* B **22** 5274
Zamolodchikov A B and Zamolodchikov A B 1979 *Ann. Phys., NY* **120** 253